Solution Estimates for Stable Linear Neutral-Type Differential–Difference Equations

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Introduction

This paper considers linear stationary differential–difference equations of neutral type [1, 2, 3, 4]. The main objective is to construct upper bounds for their solutions using the second Lyapunov method [4, 5, 6, 7, 8] and Lyapunov–Krasovskii functionals [4, 8, 10]. Stability problems of dynamical systems have been studied extensively.
A. M. Lyapunov established the classical theory of stability of motion under perturbations of initial data. Later, the concept of stability under constant and stochastic disturbances was developed [11, 12]. In the late 20th century, interest grew in systems with interval parameters, leading to interval stability and Kharitonov's theorems [13, 14, 15].
However, many results for ordinary differential equations are not directly applicable to functional–differential systems with delays [16, 17, 18]. Neutral-type systems, where delayed terms appear in derivatives, more accurately describe real processes [19]. Despite growing attention, there are still relatively few results in this area [21, 22, 23].
In this work, we study dynamical systems described by linear neutral-type differential–difference equations [1, 2, 3]. The second Lyapunov method is applied in its "coarse" form: if the derivative of the Lyapunov or Lyapunov–Krasovskii functional is negative definite, then a whole class of perturbed systems is stable [4, 8, 10].

Neutral type equations. Estimation of solutions dynamics

Constructive estimates of perturbations for linear stationary differential-difference equations of neutral type are obtained using the Lyapunov–Krasovskii functional method.
Consider the linear neutral-type equation with constant coefficients

$$\frac{d}{dt}\big[x(t) - d\,x(t-\tau)\big] = f\big(x(t), x(t-\tau)\big), \tag{1}$$

where $t \ge 0$, $\tau > 0$, and $x(t) \in \mathbb{R}$. The initial conditions are

$$x(t) = \varphi(t), \quad \dot{x}(t) = \varphi'(t), \quad -\tau \le t \le 0,$$

where $\varphi(t)$ is a continuously differentiable function. The solution $x(t)$ is piecewise continuously differentiable and may have first-kind derivative jumps at points $t = k\tau$.
We aim to estimate the deviation of $x(t)$ from equilibrium on the interval

$$-\tau \le t \le T_N, \quad T_N = m_N \tau.$$

The following norms are used:

$$\begin{aligned} \|x(t)\|_\tau &= \max_{-\tau \le s \le 0} |x(t+s)|, \\ \|x(t)\|_{\tau,\beta} &= \left(\int_{t-\tau}^{t} e^{-\beta(t-s)} x^2(s)\, ds \right)^{1/2}, \\ \|\dot{x}(t)\|_{\tau,\beta} &= \left(\int_{t-\tau}^{t} e^{-\beta(t-s)} \dot{x}^2(s)\, ds \right)^{1/2}. \end{aligned} \tag{2}$$

Because the equation is of neutral type, derivative discontinuities at nodal points are allowed, and integrals are taken in the limiting sense.
To estimate perturbations, the Lyapunov–Krasovskii functional method is applied [4, 8, 10]. A basic form of the functional is

$$V[x(t)] = [x(t) - d\,x(t-\tau)]^2 + \int_{t-\tau}^{t} g\, x^2(s)\, ds,$$

where $g > 0$ is chosen to ensure negative definiteness of its derivative along system trajectories. This functional proves asymptotic stability, but to obtain explicit estimates, a quadratic functional depending on both x and $\dot{x}$ is required.
The zero solution of a neutral-type equation is said to be exponentially stable in metric C^0 if there exist $N_1, N_2 > 0$ and $\gamma > 0$ such that for all $t > 0$

$$|x(t)| \le [N_1 \|x(0)\|_\tau + N_2 \|\dot{x}(0)\|_\tau] e^{-\frac{1}{2}\gamma t}.$$

Here $\dot{x}(0)$ denotes the right-hand derivative.
The zero solution is exponentially stable in metric C^1 if it is stable in C^0 and there exist $R_1, R_2 > 0$ and $\varsigma > 0$ such that for all $t > 0$

$$|\dot{x}(t)| \le [R_1 \|x(0)\|_\tau + R_2 \|\dot{x}(0)\|_\tau] e^{-\frac{1}{2}\varsigma t}.$$

Estimates for a Stable Scalar Neutral Equation

Model.

$$\frac{d}{dt}\big[x(t) - d\,x(t-\tau)\big] \;=\; -a\,x(t) + b\,x(t-\tau), \qquad t \ge 0,$$

with history on $[-\tau, 0]$ and norms $\|x(t)\|_\tau = \max_{-\tau \le s \le 0} |x(t+s)|$, $\|x(t)\|^2_{\tau,\beta} = \int_{t-\tau}^{t} e^{-\beta(t-s)} x^2(s)\, ds$.
Lyapunov–Krasovskii functional.

$$V_0[x(t)] = x^2(t) + \int_{t-\tau}^{t} e^{-\beta(t-s)} \big(g_1 x^2(s) + g_2 \dot{x}^2(s)\big)\, ds, \quad g_1, g_2, \beta > 0.$$

Feasibility condition. Define

$$S[\beta, g_1, g_2] = \begin{bmatrix} 2a - g_1 - a^2 g_2 & -b(1-ag_2) & -d(1-ag_2) \\ -b(1-ag_2) & e^{-\beta\tau} g_1 - b^2 g_2 & -bdg_2 \\ -d(1-ag_2) & -bdg_2 & (e^{-\beta\tau} - d^2) g_2 \end{bmatrix}.$$

If $S[\beta, g_1, g_2] \succ 0$, then exponential estimates follow.
Main estimate There exist $\gamma, \varsigma > 0$ such that

$$|x(t)| \;\le\; \Big[(1 + \tau\sqrt{g_1}) \|x(0)\|_\tau + \tau\sqrt{g_2}\, \|\dot{x}(0)\|_\tau\Big] e^{-\frac{\gamma}{2} t},$$

$$\|\dot{x}(t)\| \;\le\; \Big[\tfrac{|b|}{|d|} + M(1 + \tau\sqrt{g_1})\Big] \|x(0)\|_\tau + \big[1 + M\tau\sqrt{g_2}\big] \|\dot{x}(0)\|_\tau\, e^{-\frac{\varsigma}{2} t},$$

where

$$\gamma = \min\{\, 2\lambda_{\min}(S[\beta, g_1, g_2]),\ \beta\,\}, \qquad \varsigma = \min\Big\{ \tfrac{2}{\tau} \ln \tfrac{1}{|d|},\ \gamma \Big\},$$

and for $t \in [(m-1)\tau, m\tau)$,

$$M = \frac{|ad - b|}{1 - |d| e^{\frac{\gamma}{2}\tau}}\, e^{\frac{\gamma}{2}\tau} \Big[1 - (|d| e^{\frac{\gamma}{2}\tau})^{m-1}\Big].$$

Quick guidance. Typical feasibility: $a > 0$, $|d| < 1$, and *moderate* $|b| \Rightarrow S \succ 0$ more likely. Larger a, smaller $|d|$ or $|b|$, and appropriate β improve decay rates.

Discussion and Conclusions

As already noted by the authors, it is generally accepted that the widely known and frequently used direct Lyapunov method is still in some sense a "rough" method. In the sense that it quite simply gives an answer about the stability or instability of solutions of the studied dynamic systems. But at the same time, there is often no answer to the question of what laws exactly cause the damping or scattering of these solutions. This is especially evident in the case when the dynamics of the process are described in terms of functional-differential equations of neutral type, regardless of which variation of the direct Lyapunov method is used by researchers: the Lyapunov-Krasovskii functional approach or Lyapunov functions with additional conditions of the Razumikhin type [2]. In this article, the authors, having chosen the Lyapunov-Krasovskii functional approach, demonstrated in detail a technique using a simple neutral equation as an example in the proof of the main theorem, which not only establishes the fact of stability of solutions, but most importantly, shows that the decay of the norms of the solution and its derivative occurs according to the exponential law. Obviously, the presented methodology is quite constructive. By applying it in the future, it is possible to similarly obtain results on the exponential decay of solutions of more complex neutral systems, both linear and with nonlinearities of a certain type. Also, thanks to the presented approach, it is possible to obtain similar constructive results for discrete systems described by difference equations with argument deviation.

References

[1] R. Bellman, K. Cooke, *Differential-Difference Equations*, *Academic Press*, New York - London, 1963.

[2] J. K. Hale, *Theory of Functional Differential Equations*, *Springer-Verlag*, 1997.

[3] V. Kolmanovskii, A. Myshkis, *Applied Theory of Functional Differential Equations*, *Kluwer Academic Publishers*, Dordrech-Boston-London, 1992.

[4] D. Ya. Khusainov, A. V. Shatyrko, *Method of Lyapunov Functions in the Study of Stability of Differential-Functional Systems*, *Kyiv University Press*, Kyiv, 1997, 236 p.

[5] A. M. Lyapunov, *The General Problem of the Stability of Motion*, *State Publishing House of Technical and Theoretical Literature*, Moscow–Leningrad, 1950.

[6] A. M. Lyapunov, *The General Problem of the Stability of Motion*, De Gruyter Series in Nonlinear Analysis and Applications: 4, *International Journal of Control*, **55**(3) (1992), 531–534.

[7] A. A. Martynyuk, *Stability of Motion: The Role of Multicomponent Liapunov's Functions*, *Cambridge Scientific Publishers*, 2007.

[8] N. N. Krasovskii, *Stability of Motion*, Ed. by J. L. Brenner, *Stanford University Press*, 1963.

[9] D. Khusainov, A. Shatyrko, T. Shakotko, R. Mustafaeva, An optimization approach to constructing Lyapunov-Krasovskii functionals, *Journal of Numerical and Applied Mathematics*, **1**(2) (2023), 165–173. doi:10.17721/2706-9699.2022.2.19

[10] M.-J. Park, O. M. Kwon, J. H. Park, S.-M. Lee, A new augmented Lyapunov-Krasovskii functional approach for stability of linear systems with time-varying delays, *Applied Mathematics and Computation*, **217**(17) (2011), 7197–7209.

[11] I. A. Dzhalladova, D. Ya. Khusainov, Convergence estimates for solutions of a linear neutral type stochastic equation, *Functional Differential Equations*, **18**(3–4) (2011), 177–186.

[12] X. Wang, J. Xia, J. Wang, Z. Wang, J. Wang, Reachable set estimation for Markov jump LPV systems with time delays, *Applied Mathematics and Computation*, **376** (2020), 125117.

[13] V. L. Kharitonov, *Time-Delay Systems: Lyapunov Functionals and Matrices*, *Springer Science & Business Media*, 2012.

[14] A. G. Mazko, *Matrix Equations, Spectral Problems and Stability of Dynamic Systems*, *Cambridge Scientific Publishers*, 2008.

[15] Q.-L. Han, Robust stability of uncertain delay-differential systems of neutral type, *Automatica*, **38** (2002), 719–723.

[16] A. Seuret, F. Gouaisbaut, L. Baudouin, D1.1 – Overview of Lyapunov methods for time-delay systems, [Research Report] Rapport LAAS n°16308, LAAS-CNRS, 2016. hal-01369516

[17] P. G. Park, W. I. Lee, S. Y. Lee, Auxiliary function-based integral inequalities for quadratic functions and their applications to time-delay systems, *Journal of the Franklin Institute*, **352**(4) (2015), 1378–1396.

[18] O. M. Kwon, S. H. Lee, M. J. Park, S. M. Le, Augmented zero equality approach to stability for linear systems with time-varying delay, *Applied Mathematics and Computation*, **381** (2020). doi:10.1016/j.amc.2020.125329

[19] M. S. Mahmoud, Recent progress in stability and stabilization of systems with time-delays, *Mathematical Problems in Engineering*, **2017**, Article ID 7354654, 25 pp. doi:10.1155/2017/7354654

[20] M. Liu, Stability analysis of neutral-type nonlinear delayed systems: An LMI approach, *Journal of Zhejiang University SCIENCE A*, **7** (2006), 237–242.

[21] D. Khusainov, A. Shatyrko, R. Mustafaeva, Estimations of solutions to unstable linear differential-difference equations of neutral type, *Bulletin of the Institute of Mathematics*, **7**(6) (2024), 69–75.

[22] Q.-L. Han, A discrete delay decomposition approach to stability of linear retarded and neutral systems, *Automatica*, **45** (2009), 517–524.

[23] J. Diblík, D. Ya. Khusainov, A. Shatyrko, J. Baštinec, Z. Svoboda, Absolute stability of neutral systems with Lurie type nonlinearity, *Journal of Advanced Nonlinear Analysis*, **11** (2021), 726–740. doi:10.1515/anona-2021-0216