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Contents

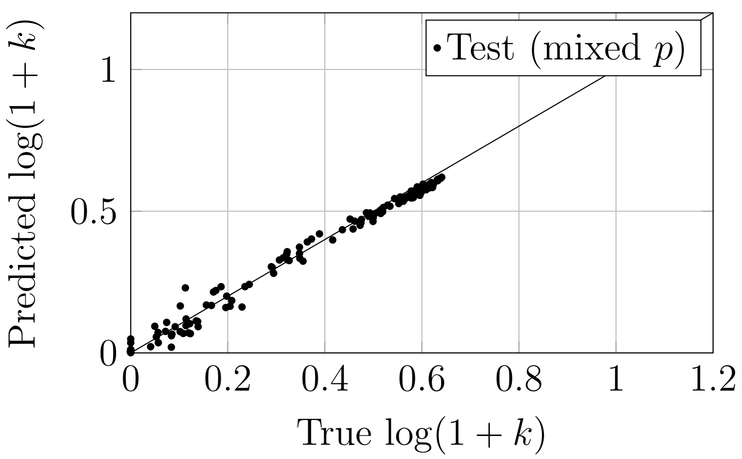
Graph neural networks for permeability estimation in porous media

Oleg Afanasiev (*Kharkiv, Ukraine*)

We demonstrate a working graph–neural surrogate that predicts the effective permeability k_{eff} of random porous structures with physics-consistent targets. Binary Bernoulli percolation on 24×24 grids is used to generate pore layouts. Open cells form nodes and undirected 4-neighborhoods form edges. The "true" k_{eff} is obtained by a discrete Darcy/Laplace solve with Dirichlet boundary conditions ($P = 1$ left, $P = 0$ right); components not touching the boundaries are discarded; if there is no left–right connectivity, $k_{\text{eff}} = 0$.

A lightweight two-layer GCN with sparse message passing and sum pooling is trained on $z = \log(1 + k_{\text{eff}})$ with a Huber loss ($\delta = 0.1$) and early stopping by $R^2_{\log k}$. Node features comprise degree, normalized coordinates $(x/W, y/H)$, global porosity, local porosity (5×5 average), left/right distances, and normalized graph distance to the right boundary. Training is stratified over porosity p: base interval $[0.40, 0.80]$ with additional slices $[0.30, 0.40]$, $[0.80, 0.90]$, and $[0.90, 0.97]$.

Results. On a stratified test ($N \approx$ 1k graphs), the model attains $R^2_{\log k} \approx 0.993$, $\text{RMSE}_k \approx 0.028$, and $\text{MAE}_k \approx 0.019$. For low-$p$ extrapolation (0.20–0.35) the errors are small ($\text{MAE}_k \approx 0.0076$, $\text{RMSE}_k \approx 0.0137$; $R^2_{\log k}$ undefined due to near-zero variance). For high-p extrapolation (0.93–0.97) we obtain $R^2_{\log k} \approx 0.727$, $\text{RMSE}_k \approx 0.0240$, and $\text{MAE}_k \approx 0.0211$.



Regime	$R^2_{\log k}$	RMSE_k	MAE_k
In-range (mixed p)	**0.993**	**0.028**	**0.019**
Low-p (0.20–0.35)	n/a	0.0137	0.0076
High-p (0.93–0.97)	**0.727**	0.0240	0.0211

Conclusion. The GNN surrogate is accurate in-range and remains informative at ultra–high porosities when a small high-p tail is included during training.

Keywords: porous media; permeability; percolation; graph neural networks; physics-informed ML.

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On output stabilization of nonlinear systems

Maksym Bebiya (*Kharkiv, Ukraine*)

We consider the stabilization problem for a nonlinear system of the form

$$\begin{cases} \dot{x}_1 = x_2 + f_1(x_1), \\ \dot{x}_2 = u + f_2(x_1, x_2), \\ y = x_1, \end{cases} \tag{1}$$

where $u \in \mathbb{R}$ is a control input, y is the measurable output, $f_1(x_1)$ and $f_2(x_1, x_2)$ are continuous functions with $f_1(0) = 0$, $f_2(0,0){=}0$.

Our objective is to achieve global asymptotic stabilization of system (1) using the available information about its state. An output-feedback control law is derived in the form

$$\begin{cases} \dot{z} = \varphi(z, y), \quad z \in \mathbb{R}, \\ u = u(z, y). \end{cases}$$

To implement this output-feedback control, we first design a full-state stabilizing feedback law. Next, we construct a reduced-order observer [1] to estimate the unmeasured component of the state vector. This estimate is then used in place of the corresponding actual state to implement the output-feedback control. The full-state feedback can be constructed recursively [2] or nonrecursively [3]. The recursive technique is based on the backstepping method that provides a systematic procedure for constructing stabilizing control laws together with corresponding Lyapunov functions. In contrast, nonrecursive approaches rely on direct Lyapunov technique, which may yield simpler control laws but is generally less structured and more problem-specific.

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The least-squares method in the theory of nonlinear boundary value problems with delay

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Viktor Chuiko (*Kyiv, Ukraine*)

The present work addresses the issue of solution existence

$$z(t,\varepsilon): \; z(\cdot,\varepsilon) \in \mathbb{C}^1[0,T], \; z(t,\cdot) \in \mathbb{C}[0,\varepsilon_0]$$

for nonlinear periodic boundary value problem with delay [1]

$$dz(t,\varepsilon)/dt = A(t)z(t,\varepsilon) + B(t)z(t-\Delta,\varepsilon) + f(t) + \\ +\varepsilon\, Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), t, \varepsilon) \quad (1)$$

in a small neighborhood of the solution $z_0(t) \in \mathbb{C}^1[0,T]$ of the generating problem

$$dz_0(t)/dt = A(t)z_0(t) + B(t)z_0(t-\Delta) + f(t), \quad \Delta \in \mathbb{R}^1. \quad (2)$$

Here $A(t), B(t)$ are $(n\times n)$-dimensional matrices, $Z(z(t,\varepsilon), z(t-\Delta,\varepsilon), t, \varepsilon)$ is a nonlinear vector function, T-periodic to the independent variable t, analytic with respect to the unknown $z(t,\varepsilon)$ and $z(t-\Delta,\varepsilon)$ in a small neighborhood of the solution of the generating problem (2) and continuous with respect to the small parameter ε on the segment $[0,\varepsilon_0]$. In addition, the function $f(t)$ is continuous with respect to the $t \in [0,T]$.

The relevance of the study of the boundary value problem (1) is related to the wide application of similar problems in the study of non-isothermal chemical reactions [2]. As is known, in the critical case, namely, in the presence of T – periodic solutions $z_0(t,c_r) = X_r(t)c_r, \; c_r \in \mathbb{R}^r$ of the homogeneous part

$$dz_0(t)/dt = A(t)\, z_0(t) + B(t)z_0(t-\Delta) \quad (3)$$

of the system (2), and in the case of constant matrices $A(t) \equiv A$ and $B(t) \equiv B$, in the presence of purely imaginary roots of the characteristic equation the generating periodic problem for the equation (2) is not solvable for all vector functions $f(t)$. In the critical case, the adjoint system [3]

$$dy(t)/dt = -A^*(t)y(t) - B^*(t)y(t+\Delta)$$

has a family of T – periodic solutions of the form $y(t,c_r) = H_r(t)c_r, \; c_r \in \mathbb{R}^r$. The periodic problem for equation (2) is solvable under the condition [3]

$$\int_0^T H_r^*(s)\, f(s)\, ds = 0. \quad (4)$$

Here, $H_r(t)$ is an $(n \times r)$ – dimensional matrix formed from r linearly independent T – periodic solutions of the adjoint system. Let us assume that condition (4) is satisfied; in this case, the general solution of the generating T – periodic problem for the equation (2) takes the form

$$z_0(t, c_r) = X_r(t)\, c_r + G[f(s)](t), \quad c_r \in \mathbb{R}^r,$$

where $G[f(s)](t)$ is a particular solution of the generating T – periodic problem for the equation (2) with delay, $X_r(t)$ is an $(n \times r)$ – dimensional matrix formed by r linearly independent T – periodic solutions of the system (2). As is known [1, 3], if, for the generating periodic problem for the equation (2), a critical case occurs, and the T – periodic problem for the equation (1) has a T – periodic solution that, at $\varepsilon = 0$, transforms into the generating solution $z(t, 0) = z_0(t, c_r^*)$, then the vector $c_r^* \in \mathbb{R}^r$ satisfies the equation for generating amplitudes

$$F(c_r^*) := \int_0^T H_r^*(s) Z(z_0(s, c_r^*), z_0(s - \Delta, c_r^*), s, 0)\, ds = 0. \tag{5}$$

By applying the Adomian decomposition method and the least squares method scheme, we have derived the necessary and sufficient conditions for the existence of solutions to the weakly nonlinear periodic boundary value problem for a system of differential equations with concentrated delay in the critical case. The efficiency of the iterative schemes we have developed is demonstrated using an example of solving the problem of approximating periodic solutions to an equation with concentrated delay, which models a non-isothermal chemical reaction [2].

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Weakly nonlinear periodic boundary value problems with switchings

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Olga Nesmelova (*Sloviansk, Cherkasy, Ukraine*)

We study the problem of constructing solutions [1, 2]

$$z(\cdot,\varepsilon) \in \mathbb{C}^1\{[0,T] \setminus \{\tau(\varepsilon)\}_I\} \cap \mathbb{C}[0,T], \quad z(t,\cdot) \in \mathbb{C}[0,\varepsilon_0]$$

of the autonomous boundary value problem for the equation

$$z'(t,\varepsilon) = A\,z(t,\varepsilon) + \varepsilon\,Z(z(t,\varepsilon),\varepsilon), \quad \ell z(\cdot,\varepsilon) = 0, \tag{1}$$

which continuous at $t = \tau(\varepsilon)$. At the point $t = \tau(\varepsilon)$:

$$0 < \tau(\varepsilon) < T, \quad \tau(0) := \tau_0$$

the solution of the boundary value problem (1) might have a limited discontinuity of first derivative [1, 2]. The solution of the boundary value problem (1) is found in a small neighbourhood of the solution

$$z_0(t) \in \mathbb{C}\Big\{[0,T] \setminus \{\tau_0\}_I\Big\} \cap \mathbb{C}[0,T]$$

of the generating boundary value problem

$$z_0'(t) = A\,z_0(t), \quad \ell z_0(\cdot) = 0, \tag{2}$$

At the point $t = \tau_0$ the solution of the boundary value problem (2) might have a limited discontinuity of the derivative. Where, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, $Z(z,\varepsilon)$ is a nonlinear vector function, piecewise analytic in the unknown z in a small neighbourhood of the solution of the generating problem (2) and piecewise analytic in a small parameter ε on the interval $[0,\varepsilon_0]$. In addition,

$$\ell z(\cdot,\varepsilon) := \begin{pmatrix} z(0,\varepsilon) - z(T,\varepsilon) \\ z(\tau(\varepsilon)+0,\varepsilon) - z(\tau(\varepsilon)-0,\varepsilon) \end{pmatrix} = 0$$

and

$$\ell z_0(\cdot) := \begin{pmatrix} z_0(0) - z_0(T) \\ Z_0(\tau_0+0) - z_0(\tau_0-0) \end{pmatrix} = 0$$

are linear bounded vector functionals.

The autonomous boundary value problem for the system (1) with switchings at non-fixed pionts in time on a fixed length interval is significantly different from similar boundary value problems with switchings at fixed points in times [3].

We are found the constructive conditions of solvability and the scheme for constructing solutions of the nonlinear periodic boundary value problem with switchings at non-fixed points in time.

Using the Adomian decomposition method [4, 5, 6], the solvability conditions are obtained and a new iterative technique for finding solutions to the weakly nonlinear periodic boundary value problem with switchings at non-fixed points in time is constructed. In addition, we obtained the constructive conditions for the convergence of the iterative scheme to the solution of the weakly nonlinear boundary value problem, as well as the switchings pionts. The obtained iterative scheme is applied to find approximations to the periodic solution of the equation with switchings at non-fixed points in time, which models a nonisothermal chemical reaction [3, 6].

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The controllability function for MIMO control systems via matrix distributions

Abdon Choque-Rivero (*Morelia, Mexico*)

We introduce a formulation of Korobov's controllability function for multiple-input multiple-output (MIMO) linear control systems in terms of matrix distributions.

Let A and B be constant real matrices of dimensions $n \times n$ and $n \times r$, respectively. For a completely controllable linear MIMO state equation

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \Omega \subset \mathbb{R}^r,$$

we construct a family of bounded positional controls $u = u(x, \Theta(x))$ that solve the synthesis problem for the system. We employ the controllability function $\Theta(x)$ [1, 2], defined as the positive solution of the implicit equation

$$2a_0\Theta = \big(N^{-1}(\Theta)x,\, x\big), \qquad a_0 > 0,$$

where

$$N(\Theta) = \int_0^\infty \Theta e^{-At} B\, d\sigma\left(\tfrac{t}{\Theta}\right)\, B^* e^{-A^*t},$$

with positive parameter Θ. Here, $\sigma(s)$ is a positive matrix distribution, that is, an $r \times r$ nondecreasing matrix-valued function of bounded variation on $[0, \infty)$ [3, 4].

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Adomian decomposition method for the nonlinear periodic boundary-value problems

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We study the problem of constructing of an analytic solutions

$$z(t) \in \mathbb{C}^1[0, T]$$

of the nonlinear periodic boundary-value problem [1, 2]

$$dz/dt = A\,z + f(t) + Z(z, t), \quad \ell z(\cdot) := z(0) - z(T) = 0 \tag{1}$$

in a small neighborhood of the analytic solution of the generating problem

$$dz_0/dt = A\,z_0, \quad \ell z_0(\cdot) := z_0(0) - z_0(T) = 0. \tag{2}$$

where, A is a constant $(n \times n)$-dimensional matrix, $Z(z, t)$ is a nonlinear vector function analytic in the unknown z in a small neighborhood of the solution of the generating problem (2). In addition, the vector function $Z(z, t)$ and the function $f(t)$ are continuous in the independent variable t on the segment $[0, T]$. In the critical case

$$\det Q = 0, \quad Q := \ell X(\cdot)$$

and the generating problem (2) under the condition [1]

$$P_{Q_r^*} \ell K[f(s)](\cdot) = 0. \tag{3}$$

has an r parameter family of solutions

$$z_0(t, c_r) = X_r(t)\,c_r + G[f(s)](t), \quad c_r \in \mathbb{R}^r.$$

Where, $X(t)$ is a normal $(X(0) = I_n)$ fundamental matrix of the homogeneous part of the differential system (2) and the matrix $X_r(t)$ consists of r linearly independent columns of the normal fundamental matrix $X(t)$. The matrix $P_{Q_r^*}$ is formed by r linearly independent rows of the matrix orthoprojector

$$P_{Q^*} : \mathbb{R}^n \to \mathbb{N}(Q^*).$$

Furthermore,

$$G[g(s)](t) = K[g(s)](t) - X(t)Q^+ \ell K[g(s)](\cdot)$$

is the generalized Green operator of the periodic boundary-value problem

$$\frac{dy}{dt} = A\,y + g(t), \quad y(0) - y(T) = 0$$

in the critical case and Q^+ is the pseudoinverse Moore–Penrose matrix. We also consider the Green operator [1, 3]

$$K[g(s)](t) := X(t) \int_a^t X^{-1}(s)\, g(s)\, ds, \quad t \in [a, b]$$

of the Cauchy problem

$$dy/dt = A\, y + g(t), \quad y(a) = 0.$$

Where, $g(t) \in \mathbb{C}[a, b]$ is a continuous vector function. It is known that the critical case occurs if and only if the matrix A has eigenvalues on the imaginary axis.

For the nonlinear periodic boundary-value problem posed for an ordinary differential equation (2) in the critical and noncritical cases, we obtain constructive conditions of its solvability and propose a scheme for finding its solutions by using the Adomian decomposition method [4, 5, 6]. To illustrate the efficiency of the proved theorem, we consider the problem of determination of analytic solutions of the nonlinear Duffing equation with perturbation.

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Parabolic De Giorgi classes with doubly nonlinear, nonstandard growth

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We define a suitable class $\mathcal{PDG}$ of functions bearing unbalanced energy estimates, that are embodied by local weak subsolutions to doubly nonlinear, double-phase, Orlicz-type and fully anisotropic operators. Yet we prove that members of $\mathcal{PDG}$ are locally bounded, under critical, sub-critical and limit growth conditions typical of singular and degenerate parabolic operators, with quantitative point-wise estimates that follow the lines of the pioneering work of Ladyzhenskaya, Solonnikov and Uraltseva [1]. These local bounds are new in the critical and sub-critical cases, and have been obtained without a qualitative boundedness assumption. In particular, the proof of local boundedness in the critical case covers both the classical p-Laplacian and the porous medium equations.

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On approximate controllability problems for the heat equation in a half-plane controlled by the Dirichlet boundary condition with a bounded control

Larissa Fardigola (*Kharkiv, Ukraine*)
Kateryna Khalina (*Kharkiv, Ukraine*)

We consider the control system

$$w_t = \Delta w, \qquad x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{R},\ t \in (0,T), \tag{1}$$
$$w\big(0,(\cdot)_{[2]},t\big) = u((\cdot)_{[2]},t), \qquad x_2 \in \mathbb{R},\ t \in (0,T), \tag{2}$$
$$w\big((\cdot)_{[1]},(\cdot)_{[2]},0\big) = w^0, \qquad x_1 \in \mathbb{R}_+,\ x_2 \in \mathbb{R}, \tag{3}$$

where $\mathbb{R}_+ = (0,+\infty)$, $T > 0$, $\Delta = (\partial/\partial x_1)^2 + (\partial/\partial x_2)^2$, $u \in U[0,T]$ is a control,

$$U[0,T] = \left\{ \varphi \in L^\infty(\mathbb{R} \times (0,T)) \,\middle|\, \sup_{t\in[0,T]} |\varphi(\cdot,t)| \in L^2(\mathbb{R}) \right\}$$

is the set of admissible controls. The subscripts [1] and [2] associate with the variable numbers, e.g. $(\cdot)_{[1]}$ and $(\cdot)_{[2]}$ correspond to x_1 and x_2, respectively, if we consider $f(x)$, $x \in \mathbb{R}^2$. This problem is considered in spaces of the Sobolev type.

Controllability problems for the heat equation were studied both in bounded and unbounded domains. However, most of the papers studying these problems deal with domains bounded with respect to the spatial variables.

For a bounded domain $\Omega \subset \mathbb{R}^n$ with the boundary $\partial\Omega$ of class C^2 (which is considered instead of the domain $\mathbb{R}_+ \times \mathbb{R}$), it is well-known that the control system of the form (1)–(3) is null-controllable for a given time $T > 0$. This result was obtained by using Carleman inequalities (see, e.g. [1]).

For unbounded domains, the situation is essentially different. There exist pairs of initial and target states where the initial state can be driven to the end state by means of control system (1)–(3), and there exist those where the initial state cannot be driven to the end state by means of this system. For instance, there is no initial data in any negative Sobolev space that may be driven to zero in finite time (see [2]).

We study the approximate controllability problem for system (1)–(3) in Sobolev spaces under controls from $U[0,T]$, in particular, $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ and $w(\cdot,t) \in L^2(\mathbb{R}_+ \times \mathbb{R})$, $t \in [0,T]$. We show that $L^2(\mathbb{R} \times (0,T))$-controls are not appropriate to consider approximate controllability property for $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ and $w(\cdot, t) \in L^2(\mathbb{R}_+ \times \mathbb{R})$, $t \in [0, T]$, because there exists a control $u \in L^2(\mathbb{R} \times (0, T))$ (with compact supports) for which the end state $w(\cdot, T)$ of the solution to (1)–(3) does not belong to $L^2(\mathbb{R}_+ \times \mathbb{R})$ for the initial state $w^0 = 0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$. This is why we consider the narrower set of controls $U[0, T]$, which provides the condition $w(\cdot, t) \in L^2(\mathbb{R}_+ \times \mathbb{R})$, $t \in [0, T]$, for any $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$. Roughly speaking, we consider a specific subset of bounded controls in $L^2(\mathbb{R} \times (0, T))$. We prove that each initial state $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ of system (1)–(3) can be driven to an arbitrary neighbourhood of any target state $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ by choosing an appropriate control $u \in U[0, T]$, in other words, a state $w^0 \in L^2(\mathbb{R}_+ \times \mathbb{R})$ is approximately controllable to a target state $w^T \in L^2(\mathbb{R}_+ \times \mathbb{R})$ in a given time T. The method of proving this assertion is constructive. This allows to provide a numerical algorithm of solving the approximate controllability problem for system (1)–(3). To this aid, we consider the odd extension of w and w^0 with respect to x_1 and obtain a new control problem. Then we develop the state and the control in this new control system in the Fourier series with respect to a basis generated by Hermite functions that allows us to reduce the 2-d problem to a finite family of the 1-d ones. To construct controls solving them, we apply the method introduced in [3] for solving the approximate controllability problem for the 1-d heat equation controlled by the Dirichlet boundary condition. Finally, we apply the Fourier transform and its inverse to analyse the solution to control problem. The results is published in [4].

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Optimal exact observability of vibrating systems

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Jarosław Woźniak (*Szczecin, Poland*)

We consider a general class of dynamical systems with observation of the form

$$\begin{cases} \dot{Z} = \mathcal{A}Z, \\ Y = \mathcal{C}Z, \end{cases} \tag{1}$$

where $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a C_0-semigroup $\mathcal{T}(t)$, $\mathcal{C} : \mathcal{H} \to \mathbb{C}$ is a linear (unbounded) observation operator, $\mathcal{H}$ is a Hilbert space.

Under some assumptions on asymptotic spectral analysis of the differential operator of the system, we can prove that considered system is not exactly observable in the default topologies setting. Then, we show that system (1) becomes exactly observable after introducing stronger topology for state observation.

The main result of the talk is devoted to find optimal topology of the observable space. Obtained results can be applied to Timoshenko beam systems.

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Polynomial and copolynomial solutions of infinite order linear partial differential equations

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Let K be an arbitrary commutative integral domain with identity of characteristic 0, $K[x_1,...,x_n]$ is the ring of polynomials with coefficients in K and $K[x_1,...,x_n]'$ is the module of K-linear mappings from $K[x_1,...,x_n]$ to K. By a copolynomial over the ring K we mean an element of the module $K[x_1,...,x_n]'$ [1]. If $T \in K[x_1,..,x_n]'$ and $p \in K[x_1,\dots,x_n]$, then for the value of T on p we use the notation (T,p). For any multi-index $\alpha = (\alpha_1,...,\alpha_n) \in \mathbb{N}_0^n$ the derivative $D^\alpha T = \frac{\partial^{|\alpha|}T}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$ $(|\alpha| = \sum\limits_{j=1}^{n}\alpha_j)$ of a copolynomial T is defined in the same way as in the classical theory of generalized functions: $(D^\alpha T,p) = (-1)^{|\alpha|}(T,D^\alpha p), \quad p \in K[x_1,...,x_n]$. For $x = (x_1,...,x_n)$ we denote $x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$. In the ring K, we consider the discrete topology. Further, in the module of copolynomials $K[x_1,\dots,x_n]'$, we consider the topology of pointwise convergence. This topology is generated by the following metric:

$$d(T_1,T_2) = \sum_{|\alpha|=0}^{\infty}\frac{d_0((T_1,x^\alpha),(T_2,x^\alpha))}{2^{|\alpha|}},$$

where d_0 is the discrete metric on K. The convergence of a sequence $\{T_k\}_{k=1}^\infty$ to T in $K[x_1,\dots,x_n]'$ means that for every polynomial $p \in K[x_1,\dots,x_n]$ there exists a number $k_0 \in \mathbb{N}$ such that

$$(T_k,p) = (T,p), \quad k = k_0, k_0+1, k_0+2, \dots.$$

Let $\mathcal{F} = \sum\limits_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$ be a differential operator of infinite order on $K[x_1,...,x_n]$ or $K[x_1,...,x_n]'$ with coefficients $a_\alpha \in K$.

We have obtained the following main results.

Theorem 1. *Assume that* $p \in K[x_1,...,x_n]$, a_0 *is an invertible element of* K *and the formal power series* $\sum\limits_{|\alpha|=0}^{\infty} c_\alpha s^\alpha$ *is inverse for the formal power series* $\sum\limits_{|\alpha|=0}^{\infty} a_\alpha s^\alpha$ *in the ring* $K[[s_1,...,s_n]]$. *Then the polynomial* $u(x) = \sum\limits_{|\alpha|=0}^{m} c_\alpha D^\alpha p(x)$, *where* $m = \deg p$, *is the unique polynomial solution of the equation* $\mathcal{F}u = p$. *Moreover,* $\deg u = \deg p$.

Theorem 2. *The homogeneous equation* $\mathcal{F}u = 0$ *has only trivial solution in* $K[x_1, ..., x_n]$ *if and only if* $a_0 \neq 0$*.*

Theorem 3. *Let* K *be a field of characteristic 0 and let* $\mathcal{F} \neq 0$*. Then the equation* $\mathcal{F}u = p$ *has a polynomial solution for any* $p \in K[x_1, ..., x_n]$*.*

Theorem 4. *Let* K *be an integral domain of characteristic 0. Assume that* $\mathcal{F} \neq 0$*. Then the homogeneous equation* $\mathcal{F}u = 0$ *in the module* $K[x_1, ..., x_n]'$ *has only trivial solution.*

Theorem 5. *Let* K *be a field of characteristic 0. Then* $\mathrm{Im}\mathcal{F}$ *is a closed subspace in* $K[x_1, \ldots, x_n]'$*.*

Theorem 6. *Let* K *be a field of characteristic 0,* $T \in K[x_1, ..., x_n]'$ *and let* $\mathcal{F}^* = \sum\limits_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^\alpha : K[x_1, ..., x_n] \to K[x_1, ..., x_n]$*. Then the equation* $\mathcal{F}u = T$ *has a solution if and only if* $(T, p) = 0$ *for any* $p \in K[x_1, ..., x_n]$ *such that* $\mathcal{F}^*p = 0$*.*

The proofs of Theorems 3, 5 and 6 are based on the using of the theory of locally convex spaces over non-Archimedean valued fields [2].

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Generalised periodic solutions of implicit linear difference equations

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We consider the linear difference equation of mth order with constant coefficients

$$a_m w_{n+m} + a_{m-1} w_{n+m-1} + \ldots + a_1 w_{n+1} + a_0 w_n = f_n,\ a_m \neq 0,\ n \in \mathbb{N}_0, \quad (1)$$

where the coefficients $a_0, a_1, \ldots, a_m$ and all elements of the sequence $\{f_n\}_{n=0}^{\infty}$ belong to the commutative ring R. This equation is implicit, since a_m could be non-invertible. We are looking for a sequence $\{w_n\}_{n=0}^{\infty} \in R^{\mathbb{N}_0}$ that satisfies this equation.

Definition 1. *The sequence $\{f_n\}_{n=0}^{\infty}$ is called generalised periodic if for some $b_1, b_2, \ldots, b_k$ from the ring R it satisfies the recurrence relation*

$$b_0 f_n + b_1 f_{n+1} + \ldots + b_k f_{n+k} = 0, \quad n \in \mathbb{N}_0.$$

Theorem 1. *Suppose that the sequence $\{f_n\}_{n=0}^{\infty}$ is generalised periodic for some $b_0, b_1, \ldots, b_k$. Let the homogeneous equation*

$$a_m w_{n+m} + a_{m-1} w_{n+m-1} + \ldots + a_1 w_{n+1} + a_0 w_n = 0,\ a_m \neq 0,\ n \in \mathbb{N}_0$$

has only a trivial solution in $R^{\mathbb{N}_0}$. Then the solution $\{w_n\}_{n=0}^{\infty}$ of the equation (1), if it exists, must also be generalised periodic for the same $b_0, b_1, \ldots, b_k$.

This theorem allows us to find the solutions in explicit form for a wide class of such difference equations over the ring of polynomials and the ring of integers.

Using this result and some previous results on equations of the type (1) (see [1]) one can also obtain sum of a convergent series $\sum_{n=0}^{\infty} p^n f_n$ in the ring of p-adic integers $\mathbb{Z}_p$, where f_n satisfies an explicit reccurence relation.

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Solvability of first-order implicit linear difference equations over one finite commutative ring of order p^2

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Let us denote $\mathbb{Z}_m$ the residue class ring modulo m, and let p be a prime number. Consider the following ring of order p^2: $\mathcal{S}_p = \mathbb{Z}_p[t]/(t^2)$. Let $A = A_0 + A_1 t, B = B_0 + B_1 t \neq 0, F_n = F_{0,n} + F_{1,n} t \in \mathcal{S}_p$ $(n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\})$ be given elements of the ring $\mathcal{S}_p$, where $A_0, A_1, B_0, B_1, F_{0,n}, F_{1,n}$ $(n \in \mathbb{Z}_+)$ are the elements of the ring $\mathbb{Z}_p$. In the ring $\mathcal{S}_p$ consider the following first-order linear difference equation:

$$BX_{n+1} = AX_n + F_n, \quad n \in \mathbb{Z}_+. \tag{1}$$

Equation (1) is called *implicit*, if B is a non-invertible element of the ring $\mathcal{S}_p$. The following theorem establishes a solvability criterion for the equation (1).

Theorem 1. *The following assertions hold.*

1. *The equation (1) has finitely many solutions if and only if either $A_0 \neq 0$, or $B_0 \neq 0$. Moreover, the amount of solutions of the equation (1) is equal to $N = \begin{cases} 1, & A_0 \neq 0, B_0 = 0, \\ p^2, & B_0 \neq 0 \end{cases}$ and the general solution of this equation has the form*

$$X_n = \begin{cases} B^{-n} A^n X_0 + \sum\limits_{s=0}^{n-1} B^{-s-1} A^s F_{n-s-1}, \ B_0 \neq 0, \\ -A^{-1} F_n - BA^{-2} F_{n+1}, \ B_0 = 0, A_0 \neq 0, \end{cases}$$

where X_0 is an arbitrary element of $\mathcal{S}_p$ if $B_0 \neq 0$.

2. *The equation (1) has infinitely many solutions if and only if $A_0 = B_0 = 0$ and $F_{0,n} = 0$ for all $n \in \mathbb{Z}_+$. Moreover, the general solution of this equation has the form $X_n = X_{0,n} + X_{1,n} t$ $(n \in \mathbb{Z}_+)$, where $\{X_{1,n}\}_{n=0}^{\infty}$ is an arbitrary sequence at $\mathbb{Z}_p$ and the sequence $\{X_{0,n}\}_{n=0}^{\infty}$ is defined as follows:*

$$X_{0,n} = B_1^{-n} A_1^n X_{0,0} + \sum_{s=0}^{n-1} B_1^{-s-1} A_1^s F_{1,n-s-1}, \quad n \in \mathbb{N},$$

where $X_{0,0}$ is an arbitrary element of $\mathbb{Z}_p$.

3. *The equation* (1) *has no solutions if and only if* $A_0 = B_0 = 0$ *and* $F_{0,n} \neq 0$ *for some* $n \in \mathbb{Z}_+$.

For the equation (1) consider the initial condition

$$X_0 = Y_0, \tag{2}$$

where $Y_0 = Y_{0,0} + Y_{1,0}t \in \mathcal{S}_p$ and $Y_{0,0}, Y_{1,0} \in \mathbb{Z}_p$. The following theorem establishes a solvability criterion for the initial problem (1), (2).

Theorem 2. *The following assertions hold.*

1. *The initial problem* (1), (2) *has a unique solution if and only if one of the following conditions holds:*

   (a) $B_0 \neq 0$.

   (b) $B_0 = 0$, $A_0 \neq 0$ *and* $Y_0 = -A^{-1}F_0 - A^{-2}BF_1$.

   Moreover, the unique solution of the initial problem (1), (2) *has the form*

$$X_n = \begin{cases} B^{-n}A^nY_0 + \sum_{s=0}^{n-1} B^{-s-1}A^sF_{n-s-1}, & B_0 \neq 0, \\ -A^{-1}F_n - BA^{-2}F_{n+1}, & B_0 = 0, A_0 \neq 0. \end{cases}$$

2. *The initial problem* (1), (2) *has infinitely many solutions if and only if* $A_0 = B_0 = 0$ *and* $F_{0,n} = 0$ *for all* $n \in \mathbb{Z}_+$. *Moreover, the general solution of this initial problem is defined as* $X_n = X_{0,n} + X_{1,n}t$ ($n \in \mathbb{Z}_+$), *where* $X_{1,0} = Y_{1,0}$, $\{X_{1,n}\}_{n=1}^{\infty}$ *is an arbitrary sequence at* $\mathbb{Z}_p$ *and* $\{X_{0,n}\}_{n=0}^{\infty}$ *is defined as follows:*

$$X_{0,0} = Y_{0,0}, \ X_{0,n} = B_1^{-n}A_1^nY_{0,0} + \sum_{s=0}^{n-1} B_1^{-s-1}A_1^sF_{1,n-s-1}, \quad n \in \mathbb{N}.$$

3. *The initial problem* (1), (2) *has no solutions if and only if* $B_0 = 0$ *and one of the following conditions holds:*

   (a) $A_0 = 0$ *and* $F_{0,n} \neq 0$ *for some* $n \in \mathbb{Z}_+$.

   (b) $A_0 \neq 0$ *and* $Y_0 \neq -A^{-1}F_0 - BA^{-2}F_1$.

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An approximate solution of the Boltzmann equation

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The Boltzmann equation for the model of hard spheres has the form [1]

$$D(f) = Q(f, f); \tag{1}$$

$$D(f) \equiv \frac{\partial f}{\partial t} + \left(v, \frac{\partial f}{\partial x}\right), \tag{2}$$

$$Q(f, f) \equiv \frac{d^2}{2} \int_{R^3} dv_1 \int_{\Sigma} \Big|(v - v_1, \alpha)\Big| \\ \cdot \Big[f(v_1^*, t, x) f(v^*, t, x) - f(v, t, x) f(v_1, t, x)\Big] d\alpha. \tag{3}$$

We construct an approximate solution in the form

$$f(v, t, x) = \sum_{i=1}^{\infty} \varphi_i(t, x) M_i(v, t, x), \tag{4}$$

where $M_i(v, t, x)$ are the exact solutions of the equation (1)

$$D(M_i) = Q(M_i, M_i) = 0.$$

The coefficient functions $\varphi_i(t, x)$ are non-negative functions, smooth on $\mathbb{R}^4$ and

$$\|\varphi_i(t, x)\| = \sup_{(t,x)\in\mathbb{R}^4} \left(|\varphi_i(t, x)| + \left|\frac{\partial \varphi_i(t, x)}{\partial t}\right| + \left|\frac{\partial \varphi_i(t, x)}{\partial x}\right| \right) \neq 0.$$

As Maxwellians $M_i(v, t, x)$ in (4), we choose the expression that describes the motion of a gas of the "acceleration-compression" type

$$M_i(v, t, x) = \rho_i \left(\frac{\beta_i}{\pi}\right)^{3/2} \mathrm{e}^{-\beta_i (v - \overline{v}_i)^2}. \tag{5}$$

We use the uniform-integral error

$$\Delta = \sup_{(t,x)\in\mathbb{R}^4} \int_{\mathbb{R}^3} dv \Big| D(f) - Q(f, f) \Big|. \tag{6}$$

In the paper [2], we establish sufficient conditions for the coefficient functions $\varphi_i(t, x)$ and hydrodynamic parameters appearing in the distribution (4) and allowing us to make the analyzed deviation (6) arbitrarily small.

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Approximate solution of the time-optimal problem for non-autonomous linearizable control systems

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Grigory Sklyar (*Szczecin, Poland*)

In the talk, we discuss one approach to solving the time-optimal problem for nonlinear non-autonomous linearizable systems.

First results concerning linearizability conditions for nonlinear control systems were obtained in 1973 [1, 2], and later they were generalized in various directions. In particular, the case of minimal requirements for smoothness of systems was considered in [3], and the generalization of the approach to non-autonomous systems was proposed in [4] and then developed in [5, 6].

An obvious advantage of linearizable systems is that we can solve control problems for such systems using well-known methods from the linear control theory. More specifically, if a nonlinear system is linearizable, then there exists a change of variables that maps the system to a linear one. Precise definition depends on the class of considered systems (autonomous/non-autonomous systems, requirements for smoothness of systems and changes of variables, etc.) or on the domains where the systems are considered (fixed set or neighborhood). In any case, if we know this mapping, we can move to a linear system, solve the corresponding problem for it, and then return to the original system. However, in general, determining a linearizing mapping is a complex problem that typically cannot be solved explicitly. In particular, to find this mapping, one must solve a system of PDE of the first order.

Thus, it is interesting to develop approaches that allow finding a linear system without knowing the linearizing mapping; at least, without knowing this mapping in its analytical form.

In the talk, we consider the time-optimal problem for nonlinear non-autonomous systems with one-dimensional control of the form

$$\dot{x} = a(t,x) + b(t,x)u, \quad a(t,0) \equiv 0,$$
$$x(0) = x^0, \; x(\theta) = 0, \; |u(t)| \le 1, \; \theta \to \min,$$

for x^0 from a neighborhood of the origin, and show how to combine two results: (i) the method of successive approximations for linear non-autonomous systems [7, 8] based on solving the Markov power moment min-problem (with gaps), which can be analytically solved in many cases

[9, 10, 11], and (ii) the theorem on linearizability for non-autonomous nonlinear control systems, which can be efficiently applied [5, 6]. The proposed method does not require finding the linearizing mapping, namely, does not require solving the system of partial differential equations. Examples show the applicability of the method.

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Optimization of treatment strategy in the mathematical model of liver regeneration

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This work presents a mathematical model of liver regeneration aimed at identifying optimal therapeutic strategies using the principles of optimal control theory. The liver's regenerative process is represented as a multi-component dynamic system where various types of hepatocytes interact under the influence of internal and external factors. The system dynamics are modeled via modified Lotka–Volterra-type differential equations, and the optimization is performed using the method of adaptive dynamic programming (ADP). The main objective is to minimize treatment cost while achieving efficient liver recovery.

The regenerative process of the liver is modeled by the interaction of multiple hepatocyte populations: normal hepatocytes, binuclear hepatocytes, polyploid hepatocytes, hypertrophied hepatocytes, apoptotic cells, necrotic cells.

Let the generalized equations describing the dynamics of different liver cell populations be written in the following compact form [1]:

$$x_{t+1} = f(x_t, \tau_t, \lambda_t), \quad 0 \le \lambda_t \le 1, \quad x_0 = x^0, \quad x_t \in X, ; \lambda_t \in U, ; t \in \mathbb{N}, \tag{1}$$

where x_t represents the state vector of various functional liver cell types at discrete time t; $X \subseteq \mathbb{R}^n$ is the state space; $x^0 \in X$ is the initial cell distribution; λ_t is the vector of control parameters, which regulate processes such as mitosis, polyploidization, apoptosis, and stress response, $U \subseteq \mathbb{R}^m$ is the control space; τ_t is a predefined function representing the external influence, including both toxic effects and therapeutic interventions.

The system is modeled as an autonomous, controlled, deterministic dynamical system $\mathcal{S}(X, U, f)$, evolving in discrete time t, with the transition function $f(x_t, \tau_t, \lambda_t)$ explicitly defined for the liver regeneration model [1].

The goal is to find a control sequence $\lambda_{t t=0}^{T}$, that minimizes the cumulative cost of the regeneration process over a finite time horizon. The optimal control strategy minimizes the following cost functional:

$$J(\lambda_t) = \sum_{t=0}^{T} \left(\sum_{i=1}^{n} w_i x_i^2(t) + \rho |\lambda_t|^2 \right), \tag{2}$$

where $w_i \ge 0$ are weighting coefficients representing the cost or risk associated with deviations in specific liver cell populations, $\rho > 0$ is a regularization coefficient penalizing excessive intervention effort (treatment intensity).

To solve this discrete-time optimal control problem, we use the principle of optimality from dynamic programming. The Bellman value function $V(x)$ satisfies the recurrence [2]:

$$V(x_t) = \min_{\lambda_t \in U} \left(L(x_t, \lambda_t) + V(f(x_t, \tau_t, \lambda_t))\right) \tag{3}$$

where $L(x_t, \lambda_t) = \sum_{i=1}^{n} w_i x_i^2(t) + \rho|\lambda_t|^2$ is the immediate cost, $f(x_t, \tau_t, \lambda_t)$ is the state transition function for system $\mathcal{S}$ as defined in (1).

Since the explicit solution to the Bellman equation is computationally infeasible in high-dimensional settings, we apply the Adaptive Dynamic Programming (ADP) approach. ADP uses function approximation (e.g., neural networks or basis functions) to estimate the value function $V(x)$ and updates the control policy via reinforcement learning principles [3].

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Periodic Camassa–Holm equation via Fokas method

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Recently [1, 2], it has been shown that that the initial boundary value problem on a finite interval with x-periodic boundary conditions for the nonlinear Schrödinger (NLS) equation

$$iq_t + q_{xx} + 2|q|^2 q = 0$$

belongs to the class of so-called *linearizable problems*. For such problems, the solution $q(x,t)$ can be represented in terms of the solution of a Riemann–Hilbert (RH) problem, whose data — namely, the jump matrix and residue conditions — can be expressed in terms of the entries of the scattering matrix associated with the initial data.

In this work, we develop this formalism for the Camassa–Holm equation

$$\left(\sqrt{m+1}\right)_t = -\left(u\sqrt{m+1}\right)_x, \quad m = u - u_{xx}$$

which models the unidirectional propagation of shallow water waves over a flat bottom. The key idea is to employ both equations of the associated Lax pair as spectral problems simultaneously. In this approach, the x-equation of the Lax pair generates the spectral (scattering) problem corresponding to the initial data, while the t-equation leads to two additional spectral problems associated with the boundary values at the endpoints of the interval. These spectral functions are connected through a *global relation*, which encodes the compatibility between the initial and boundary data. Consequently, one can formulate a Riemann–Hilbert problem whose data (jump and residue conditions) can be expressed entirely in terms of the spectral functions associated with the initial data only.

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Solution estimates of stable linear differential-difference equations of neutral type

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This paper considers linear stationary differential–difference equations of neutral type [1, 2, 3, 4]. The main objective is to construct upper bounds for their solutions using the second Lyapunov method with Lyapunov–Krasovskii functionals [5, 6, 7]. Stability problems of dynamical systems have been studied for many decades, beginning with Lyapunov's classical theory of motion stability.

Traditional results for differential equations are often insufficient for systems with delays, where the present state depends on past values. Neutral-type equations, which include delayed derivatives, better describe many real processes but are analytically more complex [8, 9]. Modern research applies Lyapunov–Krasovskii functionals to obtain explicit solution estimates and stability criteria for such systems [11, 10].

In this work, the second Lyapunov method is employed in its direct (coarse) form: if the derivative of the Lyapunov–Krasovskii functional is negative definite, then the system remains stable for all admissible perturbations [4, 6, 7].

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Heat and mass exchange in nature and engineering: mathematical and thermodynamic optimization problems

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Heat and mass exchangers are widely used for industrial and domestic heating/cooling of air, water and other working fluids; mixing/separation, moistening/drying, ionization/deionization of fluids, etc. In nature, efficient heat/mass exchangers in mammals serves for heating/cooling and moistening/remoistening of the inhaled/exhaled air in cold and hot ambient conditions, accordingly. It was shown, geometry and operating conditions of the complex structures inside the nasal ducts of such animals are close to optimal exchangers providing minimum energy lost $\dot{W} \to \min$ and thermal inlet length $L_{th} \to \min$ at given volume $V = const$ [1], [2], [3], [4].

Here a brief classification of the main design types of the natural exchangers proposed as: (I) spiral (II) porous and (III) fractal structures with certain statistical regularities.

Mathematical multicriteria optimization problem

$$Z_h \to \min,\ Z_{th} \to \min,\ L_{th} \to \min,\ V = const, \tag{1}$$

where Z_h and Z_{th} are viscous and thermal dissipations in the system is considered based on the Lagrange multipliers (i), weight coefficients (ii) and Pareto frontiers (iii) approaches.

Thermodynamic optimization problem

$$S_{irr} \to \min,\ V = const, \tag{2}$$

where S_{irr} is the entropy production due to irreversible viscous and thermal processes is solved based on the method of Lagrange multipliers.

The solutions of (1) and (2) are compared and discussed for both natural structures and nature-inspired solutions for engineered applications from macro to micro/nano scales like chemical reactors [5].

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Unbounded stability enhancement and finite transition time

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The theory of stability, created by A. M. Lyapunov, was presented in his book "The general problem of the stability of motion", published in Kharkiv in 1892 [1]. This theory has been developed continuously to the present day. After the appearance of optimal control theory and Pontryagin's maximum principle, various aspects of control theory began to develop. The stabilization problem, which involves the construction of a control that ensures the stability of the equilibrium point, has been considered in great detail. The optimal stabilization problem is also considered, but without taking into account the control constraints. It also turned out to be possible to consider the stabilization problem over a finite time interval. The more general problem considered is reaching an arbitrary fixed point, not necessarily optimal, in a given region of the phase space. One of the challenges of this problem is to construct a control that satisfies the predefined constraints.

For this problem, the controllability function method was proposed by the author [2]. One way it can be constructed, together with the corresponding control that satisfies the given constraints in the case of a canonical system, is as follows [3]. The control is chosen in the form $u = \sum_{i=1}^{n} \frac{a_i}{\Theta^{n-i+1}(x)} x_i$, where $\Theta(x)$ is a controllability function defined by the equation

$$2a_0\Theta = (D(\Theta)FD(\Theta)x, x).$$

Here, the coefficients a_i are chosen in such a way that the system $\dot{x}_1 = x_2$, $\ldots$, $\dot{x}_{n-1} = x_n$, $\dot{x}_n = \sum_{i=1}^{n} a_i x_i$ is stable. In the proposed control, due to the convergence to zero of the function $\Theta(x)$ as $x \to 0$, the degree of stability of the system $\dot{x}_1 = x_2, \ldots, \dot{x}_{n-1} = x_n$, $\dot{x}_n = \sum_{i=1}^{n} \frac{a_i}{\Theta^{n-i+1}(x)} x_i$ increases unboundedly. This means the following. As $t \to \infty$ the degree of stability of the matrix

$$A = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ \frac{a_1}{\Theta^n(x)} & \frac{a_2}{\Theta^{n-1}(x)} & \ldots & \frac{a_n}{\Theta(x)} \end{pmatrix}$$

tends to infinity. The eigenvalues of the matrix

$$\tilde{A} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \end{pmatrix}$$

are multiplied by $\frac{1}{\Theta(x)}$. This allows reaching the origin in finite time.

Another approach to stability enhancement that allows reaching the origin under control constraints is the following. In the canonical system, the control is chosen as $u = \sum_{i=1}^{n} \frac{a_i}{\Theta - t} x_i$, where $\Theta > 0$ is a fixed constant. As $t \to \Theta$ the degree of stability of the matrix

$$\begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ \frac{a_1}{(\Theta-t)^n} & \frac{a_2}{(\Theta-t)^{n-1}} & \dots & \frac{a_n}{\Theta-t} \end{pmatrix}$$

gradually increases. The eigenvalues of the matrix $\tilde{A}$ are multiplied by $\frac{1}{\Theta-t}$. For $t = \Theta$, we reach the origin, and for a large enough value of Θ, the control satisfies the given constraints.

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Construction of control from a point onto a surface

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We consider the problem of admissible and time-optimal control for the linear system

$$\dot{x} = Ax + Bu, \; x \in \mathbb{R}^n, \tag{1}$$

from a point $x_0 \in \mathbb{R}^n$ onto a surface $G = \{x : F(x) = 0\}$ both with and without constraints on the control. In our paper, we present explicit ways to construct such controls. We consider the general case when the system (1) may not be completely controllable, but it holds that

$$L + G = \mathbb{R}^n, \tag{2}$$

where L is the linear span of the columns of Kalman's controllability matrix. The condition (2) comes from the paper [1] by V. I. Korobov, where it was proven that the system (1) is controllable on a surface G in an arbitrary time T if and only if this equality holds. Other papers related to the problems of admissible and optimal control onto a surface include [2, 3, 4].

The first step of the proposed approach consists of mapping the system (1) on the system

$$\begin{cases} \dot{y} = A_{11}y, \\ \dot{z} = A_{21}y + A_{22}z + B_1u, \end{cases} \tag{3}$$

via a linear change of variables. The system (3) has two parts, which we call the controllable and uncontrollable part. If there are no constraints, we construct the control using the known formula

$$u(t) = -B_1^* e^{-A_{22}^* t} N^{-1} \tilde{z}_0,$$

where $\tilde{z}_0$ is calculated as $\tilde{z}_0 = -e^{-A_{22}T} z_T + z_0 + \int_0^T e^{-A_{22}\tau} A_{21} y(\tau) \mathrm{d}\,\tau$.

Under control constraints $u \in \Omega$, we apply to the system (3) the controllability function method proposed by V. I. Korobov in [5] and its extensions for control to a non-equilibrium point [6, 7]. Unlike previous problems, we show that in our formulation the desired constrained control may not exist. An important case for which we show that the solution does exist is when the controllable part has the canonical form and the uncontrollable part has constrained trajectories.

The time-optimal control is constructed using the moment min-problem approach proposed by V. I. Korobov and G.M. Sklyar in [8]. The additional constraints include optimization with respect to unknown parameters $\alpha_1, \ldots, \alpha_k$, which determine the position of endpoint on the surface.

Other problems considered include the controllability problem between two surfaces, the time-optimal control problem from a bounded set onto a subspace, and the problem of maintaining a trajectory on a surface.

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Control synthesis of homogeneous approximations of nonlinear systems

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The objective of the talk is to describe computational methods of control synthesis for a certain class of nonlinear driftless control systems. Such systems are previously found to be simplifications (called homogeneous approximations) of more complicated non-linear systems that still preserve most crucial properties of the original ones like controllability. The class of systems in question have a special feed-forward form that is sufficiently easy to integrate and allows to solve concrete problems in control theory. Here we continue our research with describing the computational procedure for control synthesis as the extension of existing software libraries in Python language. We show that our approach leads to faster computation times compared to standard methods. The results are illustrated with some numerical experiments and simulations.

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Weak and strong nilpotency of distributions in action

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The first aim is to recall the notions of the weak and strong nilpotencies of vector distributions satisfying the bracket generating (Hörmander) condition. (The weak nilpotency is a modern name for the local nilpotentizability. The strong nilpotency implies the weak one, but not vice versa.)

The classical playground for discussing these notions are Goursat distributions, known for more than 120 years and featuring a rich tree of singularities. It is known since the year 2000 that the Goursat distributions are everywhere weakly nilpotent. But not everywhere strongly nilpotent, i.e., not everywhere equivalent to their nilpotent (or: homogeneous) approximations.

The second and main aim is to bring about the problem of finding all strongly nilpotent Goursat distributions. That is, equivalently, the problem of ascertaining all strongly nilpotent points in the stages of the so-called Goursat Monster Tower. A general road map to the solution has been sketched during the RIMS Sing 1 conference in Kyoto (Japan) in the year 2022. However, that road still awaits filling in its many technical details.

Experimental models of the interfacial MHD instability in two immiscible liquid layers (Review)

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Industrial production of aluminum using the Hall-Héroult process is a complex technology. And laboratory studies of magnetohydrodynamic (MHD) instabilities arising there face significant difficulties, since it is impossible to recreate the temperature regime of aluminum reduction cells under laboratory conditions. Therefore, it was important to find a substitute for cryolite and aluminum for use in laboratory experiments. In most experimental studies, the two-layer liquid system was replaced by a single-layer model [1, 2]. But this approach cannot take into account the main features, such as the small difference in the densities of the two layers and the large difference in their electrical conductivities. And only in a few studies was it possible to reproduce the MHD instability of a two-layer liquid system [3, 4]. It should be noted that similar problem associated with possible MHD instabilities arises in liquid metal batteries designed to store large amounts of electrical energy [5].

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On the controllability problem for a system of three connected tanks

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Let us consider a system of three connected tanks

$$\begin{cases} \dot{x}_1 = -x_1 + x_2, \\ \dot{x}_2 = x_1 - 3x_2 + x_3 + u, \\ \dot{x}_3 = x_2 - x_3, \\ x(0) = x^0, \ x(T) = 0. \end{cases} \tag{1}$$

where $x \in \mathbb{R}^3$, $u \in \mathbb{R}$ is a control. This system can be reduced to the form $\dot{x} = Ax + Bu$, where $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -3 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. This system is not completely controllable because rank of Kalman matrix [1] $rank(B, AB, A^2B) = 2$. Let us denote by L the linear span of the columns of Kalman matrix $L = Lin(B, AB, A^2B) = Lin \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and call it the controllability subspace. This means that $x_1^0 = x_3^0$.

The case 1. Apply the Cauchy formula

$$x(t) = e^{At}x^0 + e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau.$$

Then substitute $t = T$ and use $x(T) = 0 : x^0 = -\int_0^T e^{-A\tau}Bu(\tau)d\tau$. Let us consider the case $u(t) = const = c$. If $x_1^0 = x_2^0$ then $c = 0$. If $x_1^0 = x_2^0 + k$ then $c = \dfrac{2\sqrt{3}\, e^{2\sqrt{3}-4}k}{e^{4\sqrt{3}} - 1}$.

The case 2. To split the initial system (1) into controllable and uncontrollable part we introduce the change of variables

$$\begin{cases} y_1 = x_1 - x_3, \\ z_1 = x_2, \\ z_2 = x_1 + x_3, \end{cases} \quad \text{so we get} \quad \begin{cases} \dot{y}_1 = -y_1, \\ \dot{z}_1 = -3z_1 + z_2 + u, \\ \dot{z}_2 = 2z_1 - z_2. \end{cases} \tag{2}$$

Let control satisfy the constraint $|u| \leq 1$. Let us investigate the synthesis problem for the controllable part of this system (z_1 and z_2). Our approach is based on the Controllability Function Method proposed by V.I. Korobov in 1979. This problem consists in constructing a control in explicit form which depends on phase coordinates and steers an arbitrary initial point from a neighborhood of the origin to the origin in a finite time (settling-time function). Next we follow the paper [3] where the Controllability Function is proposed and the paper [2] where restriction on a_0 is found.

Theorem 1. *For any $z \neq 0$ define the Controllability Function $\Theta = \Theta(z)$ as the unique positive solution to the equation*

$$\begin{aligned}
&15e^{(8+2\sqrt{3})\Theta} + e^{(4+2\sqrt{3})\Theta}(2+8\Theta) + 16e^{(4+4\sqrt{3})\Theta}(-1+(-4+2\sqrt{3})\Theta)-\\
&-16e^{4\Theta}(1+(4+2\sqrt{3})\Theta + 3e^{2\sqrt{3}\Theta}(5+40\Theta+16\Theta^2) =\\
&= -24[[(-28+16\sqrt{3})e^{4\Theta} + (-28+16\sqrt{3})e^{(4+4\sqrt{3})\Theta} + 2e^{(4+2\sqrt{3})\Theta}+\\
&+(54+24\Theta)e^{2\sqrt{3}\Theta}]z_1^2 + [(20+12\sqrt{3})e^{4\Theta}+\\
&+(20-12\sqrt{3})e^{(4+4\sqrt{3})\Theta} + 2e^{(4+2\sqrt{3})\Theta} - (28+24\Theta)e^{2\sqrt{3}\Theta}]z_1z_2-\\
&-[(4+2\sqrt{3})e^{4\Theta} + (4-2\sqrt{3})e^{(4+4\sqrt{3})\Theta} + 2e^{(4+2\sqrt{3})\Theta} - (9+12\Theta)e^{2\sqrt{3}\Theta}]z_2^2],
\end{aligned} \tag{3}$$

besides, the domain of solvability synthesis problem is the ellipsoid Q defined by $Q = \{z : \Theta(z) \leq c\}$. At $z = 0$ we put $\Theta(0) = 0$.

Then in the domain Q the control

$$\begin{aligned}
&u(z) = -8e^{4\Theta}\Theta[(-28-16\sqrt{3}) + 2e^{2\sqrt{3}\Theta} + (-28+16\sqrt{3})e^{4\sqrt{3}\Theta}+\\
&+(54+24\Theta)e^{(-4+2\sqrt{3})\Theta}]z_1 - [(-10-6\sqrt{3}) - e^{2\sqrt{3}\Theta} + (-10+6\sqrt{3})e^{4\sqrt{3}\Theta}+\\
&+(21+12\Theta)e^{(-4+2\sqrt{3})\Theta}]z_2]/[-15e^{(8+2\sqrt{3})\Theta} - (2+8\Theta)e^{(4+2\sqrt{3})\Theta}+\\
&+(16-32(-2+\sqrt{3}\Theta))e^{(4+4\sqrt{3})\Theta} + (16+32(2+\sqrt{3}\Theta))e^{4\Theta}-\\
&-(15+120\Theta+48\Theta^2)e^{2\sqrt{3}\Theta}]
\end{aligned} \tag{4}$$

solves the local feedback synthesis problem for controllable part of the system (2) and satisfies the constraint $|u| \leq 1$. Besides the Controllability Function equals to the time of motion (settling-time function) from any initial point $z^0 \in Q$ to the origin.

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Virus infection model with advection–diffusion on two spatial domains

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Many viruses continue to be major global public health issues. Understanding the dynamics of viral infections and spread is crucial for developing effective prevention and control strategies.

We discuss qualitative properties of several mathematical models of viral infections. Our prime interest is in different classes of virus dynamics models with reaction-diffusion, logistic growth terms and a general nonlinear infection rate functional response.

Let $\Omega_1 \subset \mathbb{R}^3$ be a connected bounded domain (representing a susceptible organ) with a smooth boundary $\partial\Omega_1$. Let $T(t,x), T^*(t,x), V(t,x), Y(t,x)$, and $A(t,x)$ represent the densities of uninfected cells, infected cells, free virions, CTL immune cells, and antibodies at position $x \in \Omega_1$ at time t, respectively. The following system of PDEs with three distributed delay terms was investigated in [1].

$$\begin{cases} \dot{T}(t,x) = rT(t,x)\left(1-\frac{T(t,x)}{T_K}\right) - dT(t,x) - f^T(T(t,x),V(t,x)) + d^1\Delta T(t,x), \\ \dot{T}^*(t,x) = r^*T^*(t,x)\left(1-\frac{T^*(t,x)}{T^*_K}\right) + e^{-\omega h}\int_{-h}^{0} f^T(T(t+\theta,x),V(t+\theta,x))\xi^T(\theta,x,u_t)\,d\theta \\ \qquad\qquad -\delta T^*(t,x) - f^Y(T^*(t,x),Y(t,x)) + d^2\Delta T^*(t,x), \\ \dot{V}(t,x) = N\delta T^*(t,x) - cV(t,x) - f^A(V(t,x),A(t,x)) + d^3\Delta V(t,x), \quad x\in\Omega_1. \\ \dot{Y}(t,x) = e^{-\omega h}\int_{-h}^{0} f^Y(T^*(t+\theta,x),Y(t+\theta,x))\xi^Y(\theta,x,u_t)\,d\theta - \gamma Y(t,x) + d^4\Delta Y(t,x) \\ \dot{A}(t,x) = e^{-\omega h}\int_{-h}^{0} f^A(V(t+\theta,x),A(t+\theta,x))\xi^A(\theta,x,u_t)\,d\theta - bA(t,x) + d^5\Delta A(t,x) \end{cases} \tag{1}$$

The class of delay terms includes the state-selective delays [2, 3]. We consider the general functional responses $f^T(T,V), f^Y(T^*,Y))$, and $f^A(V,A)$ satisfying natural assumptions.

As usual, for a delay system [4] one denotes $u_t = u_t(\theta) \equiv u(t+\theta)$ for $\theta \in [-h,0], h > 0$. We denote $u(t) = u(t,\cdot) = (T(t,\cdot), T^*(t,\cdot), V(t,\cdot), Y(t,\cdot), A(t,\cdot))$ and add initial conditions to the delay system (1):

$$u(\theta) = \varphi(\theta) \equiv (T(\theta), T^*(\theta), V(\theta), Y(\theta), A(\theta)), \;\; \theta \in [-h,0], \tag{2}$$

or, in short, $u_0 = \varphi$.

In the mathematical literature for such a biological system it is standard to consider the no-flux boundary conditions (the Neumann B.Cs.) i.e. $\frac{\partial u(x)}{\partial n}|_{\partial\Omega_1} = 0$, where n denotes the exterior normal to the boundary $\partial\Omega_1$.

Asymptotic long-time behaviour of the solutions of (1), (2) is discussed.

We continue our study and develop a virus dynamics model, formulated on two spatial domains, which incorporates reaction–advection-diffusion processes and logistic growth. For a two-domain problem, a careful analysis of boundary conditions becomes especially important given the biological motivation.

The main result is the existence of a finite-dimensional global attractor for a dynamical system constructed in a Hilbert space. The main mathematical tool in our study of the long-time asymptotic behaviour of solutions is the *quasi-stability method* developed by I. Chueshov (for more details and definitions see [5]).

A persistence property is established, ensuring the uninfected susceptible host cells T and infected host cells T^* persist over time (in the domain Ω_1). It is important to mention that the persistence result on two spatial domains significantly differs from the corresponding one for the model on one spatial domain (see [1]). Namely, we have no persistence result for virus particles V on the domain Ω_2. The last has a clear biological background.

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Asymptotic behavior of eigenpairs of convolution type nonlocal operators

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Many mathematical biology and population dynamics models involve nonlocal diffusion corresponding to long-range interactions in a system. These models are typically described by evolution problems with convolution-type integral operators and their qualitative and quantitative properties can be obtained by studying of the corresponding spectral problems.

We consider spectral problems

$$-\frac{1}{\varepsilon^d}\int_\Omega J\Big(\frac{x-y}{\varepsilon}\Big)\kappa(x,y)\rho_\varepsilon(y)dy + a(x)\rho_\varepsilon(x) = \lambda_\varepsilon\rho_\varepsilon(x) \tag{1}$$

in a bounded domain in $\Omega \subset \mathbb{R}^d$, where $J(z) \geq 0$ is a continuous function on $\mathbb{R}^d$ decaying sufficiently fast as $|z| \to \infty$, $\kappa \in C^2(\overline{\Omega}\times\overline{\Omega})$, $\kappa > 0$ and (the potential) $a \in C^2(\overline{\Omega})$; $\varepsilon > 0$ is a scaling parameter. We study the asymptotic behavior of eigenvalues and eigenfunctions of (1) in the limit of small parameter ε.

We focus on the self-ajoint case when $J(z) = J(-z)$, $\kappa(x,y) = \kappa(y,x)$ and show that the principal eigenvalue of (1) exists for sufficiently small ε and converges to the minimum

$$m(x^*) = \min_{\overline{\Omega}} m(x), \quad \text{where } m(x) = a(x) - \kappa(x,x).$$

More precise asymptotic description is obtained when m satisfies some non-degeneracy conditions at x^*. Namely, if the minimum is strict and the point x^* is an inner point of Ω then we suppose the positiveness of Hessian and via rescaling by $\varepsilon^{1/2}$ we derive a limit differential spectral problem of the form:

$$-\mathrm{div} A\nabla\rho + \partial^2_{ij} m(x^*) z_i z_j \rho = \mu\rho \quad \text{in } \mathbb{R}^d. \tag{2}$$

We prove that

$$\lambda_\varepsilon = m(x^*) + \mu_k\varepsilon + \bar{o}(\varepsilon),$$

where μ_k are eigenvalues of (2). The case $x^* \in \partial\Omega$ is more sophisticated and we consider the situation when Ω is a polyhedron and $m(x)$ attains its strict minimum at x^* on a face of $\partial\Omega$. Without loss of generality, we assume that $x^* = 0$ and locally Ω is given by $x_1 > 0$ in a neighborhood of 0. Then the non-degeneracy condition reads: $\partial_{x_1} m(0) > 0$, $\partial^2_{x'_i x'_j} m(0)\,\xi'_i\xi'_j >$

$0\ \forall \xi' \in \mathbb{R}^{d-1} \setminus \{0\}$. Under these conditions, we establish the following asymptotic formula for the eigenvalues

$$\lambda_\varepsilon = m(0) + \Lambda_1 \varepsilon^{2/3} + (\beta + \mu_k)\varepsilon + \bar{o}(\varepsilon),$$

where Λ_1 is the principal eigenvalue of the 1D problem $-\theta\phi_0''(t) + \alpha t \phi_0(t) = \Lambda_1 \phi_0(t)$ on $\mathbb{R}_+$, $\phi_0(0) = 0$, μ_k are eigenvalues of a harmonic oscillator in $\mathbb{R}^{d-1}$. In this case, eigenfunctions have the asymptotic form $\rho_\varepsilon(x) = \phi_0(\varepsilon^{-2/3} x_1)\, v(\varepsilon^{-1/2} x') + ...$, that reveals emergence of two fine scales $\varepsilon^{2/3}$ and $\varepsilon^{1/2}$.

Global semigroup of weak solutions of the Novikov system

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In my talk I will discuss the global well-posedness of the Cauchy problem for the two-component Novikov system, which reads as follows:

$$\begin{aligned} &m_t + (uvm)_x + vu_x m = 0, && m = m(t,x),\ u = u(t,x),\ v = v(t,x),\\ &n_t + (uvn)_x + uv_x n = 0, && n = n(t,x), \\ &m = u - u_{xx},\ n = v - v_{xx}, && u, v \in \mathbb{R}, \quad t, x \in \mathbb{R}. \end{aligned} \tag{1}$$

We assume that the initial data (u_0, v_0) belongs to the metric space (Σ, d_Σ):

$$\begin{aligned} &\Sigma = \left\{(f,g) : f, g \in H^1(\mathbb{R}) \text{ and } \|f_x g_x\|_{L^2(\mathbb{R})} < \infty\right\}, \\ &d_\Sigma^2((f_1,g_1),(f_2,g_2)) = \|f_1 - f_2\|_{H^1}^2 + \|g_1 - g_2\|_{H^1}^2 \\ &\qquad\qquad + \|(\partial_x f_1)\partial_x g_1 - (\partial_x f_2)\partial_x g_2\|_{L^2}^2. \end{aligned}$$

Notice that (Σ, d_Σ) is a complete metric space, but not a linear space [4].

System (1) was introduced in [3] as a generalization of the scalar Novikov equation, which has the following form:

$$m_t + (u^2 m)_x + uu_x m = 0, \quad m = u - u_{xx}. \tag{2}$$

Observe that (1) with $u = v$ is equivalent to (2).

Applying the method of characteristics and the Bressan-Constantin approach [1], the work [2] constructs the global conservative weak solutions $(u,v)(t)$ of (1) for the initial data $(u_0, v_0) \in \left(H^1 \cap W^{1,4}\right)^2$, such that $(u,v)(t) \in \Sigma$. Since $\left(H^1 \cap W^{1,4}\right)^2 \subsetneq \Sigma$, these solutions do not preserve the regularity, and there is no hope to have a semigroup property for such solutions.

In our work [4], we address this question by revisiting the method of characteristics for the general initial data $(u_0, v_0) \in \Sigma$. Then, to construct a semigroup, we must consider the possible energy concentration of either $u_x^2\, dx$, $v_x^2\, dx$, or $\left(u_x^2 v_x^2\right) dx$. To this end, we retain additional information about the global solution $(u,v)(t)$ by considering the following positive Radon measure μ_t on $\mathbb{R}$ (here $\mu_t = \mu_t^{ac} + \mu_t^s$, where μ_t^{ac} and μ_t^s are, respectively, the absolutely continuous and singular parts of μ_t with respect to the Lebesgue measure on $\mathbb{R}$):

$$d\mu_t^{ac} = \left(u_x^2 + v_x^2 + u_x^2 v_x^2\right)(t)\, dx.$$

Moreover, we must retain certain sets $D_W(t), D_Z(t) \subset \mathbb{R}$, which are not empty if $\mu_t^s \neq 0$. Then, we can show that the map

$$\Psi_t \left(u_0, v_0, \mu_0; D_{W,0}, D_{Z,0}\right) = \left(u(t), v(t), \mu_t; D_W(t), D_Z(t)\right),$$

satisfies the semigroup property.

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Inverse scattering transform spectrum characterization of time-limited signals

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The nonlinear Schrödinger (NLS) equation

$$iq_z + q_{tt} + 2|q|^2 q = 0,$$

is an integrable nonlinear PDE, where $q(t,z)$ models a slow-varying complex electromagnetic field propagating along an optical fiber, with z being the distance along the fiber and t the retarded time. We are interested in the properties of the signals important in the framework of the "b-modulation" method, which is the nonlinear signal modulation technique that provides explicit control over the signal extent.

Namely, we provide a rigorous analysis of the spectral properties of signals in the case where the time-domain profile of a signal corresponding to the b-modulated data is finitely supported:

$$b(k) = \int_{-L}^{L} \beta(\tau) e^{ik\tau} d\tau \quad \text{with some } \beta(\tau) \in L^1(-L, L)$$

(which corresponds to a signal q such that $q(t) = 0$ for $|t| > \frac{L}{2}$) and when the bound states corresponding to specifically chosen discrete solitonic eigenvalues and norming constants, are also present. The analysis is given for the inverse problem for the Zakharov-Shabat system

$$\Phi_t(t,k) = -ik \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi(t,k) + \begin{pmatrix} 0 & q(t) \\ -\bar{q}(t) & 0 \end{pmatrix} \Phi(t,k), \tag{1}$$

which is the spectral equation form the Lax pair associated with the NLS equation. Using the Riemann-Hilbert problem formalism for the inverse scattering problem for (1), we prove that the number of solitary modes that we can embed without violating the exact localization of the time-domain profile, is infinite, and present the characterization of the set of all possible solitonic eigenvalues using the Riemann-Hilbert problem formalism.

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One class of C_0-groups with sub-linear growth on dense subsets

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We consider a special class of linearly growing C_0-groups from [1], whose generators are essentially nonselfadjoint unbounded operators, i.e. they are not similar to any selfadjoint operator. More precisely, let $\{e_n\}_{n=1}^{\infty}$ be a Riesz basis of a separable Hilbert space H. Then

$$H_1(\{e_n\}) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n : \ \{c_n - c_{n-1}\}_{n=1}^{\infty} \in \ell_2, \ c_0 = 0 \right\},$$

is a Hilbert space of all formal series $(\mathfrak{f}) \sum\limits_{n=1}^{\infty} c_n e_n$, and $\{e_n\}_{n=1}^{\infty}$ is complete and minimal sequence in space $H_1(\{e_n\})$ (H_1 for short) but does not form a Schauder basis of H_1, as it was shown in [1].

Furthermore [1], for each $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$, where

$$\mathcal{S}_1 = \left\{ \{f(n)\}_{n=1}^{\infty} \subset \mathbb{R} : \lim_{n\to\infty} f(n) = +\infty; \{n\,(f(n) - f(n-1))\}_{n=1}^{\infty} \in \ell_\infty \right\},$$

the operator $A_1 : H_1 \supset D(A_1) \mapsto H_1$, defined by

$$A_1 x = A_1 \left((\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \right) = (\mathfrak{f}) \sum_{n=1}^{\infty} i f(n) \cdot c_n e_n,$$

with domain

$$D(A_1) = \left\{ x = (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n \in H_1 : \ \{f(n) \cdot c_n - f(n-1) \cdot c_{n-1}\}_{n=1}^{\infty} \in \ell_2 \right\},$$

generates the following C_0-group on H_1,

$$e^{A_1 t} x = e^{A_1 t} (\mathfrak{f}) \sum_{n=1}^{\infty} c_n e_n = (\mathfrak{f}) \sum_{n=1}^{\infty} e^{itf(n)} c_n e_n, \quad t \in \mathbb{R}. \tag{1}$$

Assume that there exists a constant $K > 0$ such that $\forall n \in \mathbb{N}$ we have

$$n\,|f(n) - f(n-1)| \geq K. \tag{2}$$

Then, as it was proved in [2], [3], the C_0-group $\left\{e^{A_1 t}\right\}_{t\in\mathbb{R}}$ has an exact linear growth, i.e. there exists a linear function $\mathfrak{l}$, with positive coefficients, and constant $C > 0$ such that for all $t \in \mathbb{R}$ we have

$$C|t| \leq \left\| e^{A_1 t} \right\| \leq \mathfrak{l}(|t|). \tag{3}$$

In a recent work are taking the next step in the asymptotic analysis of this class of C_0-groups, i.e. the study of asymptotics of the norm of this C_0-group on certain dense subsets of the phase space H_1.

Theorem 1. *Let $\left\{e^{A_1 t}\right\}_{t\in\mathbb{R}}$ be the C_0-group as above, defined on the space H_1, where $\{f(n)\}_{n=1}^{\infty} \in \mathcal{S}_1$ and is monotonic. Let $k \in \mathbb{N}\cup\{0\}$ and assume that there exists a constant $K > 0$ such that $\forall n \in \mathbb{N}$ we have*

$$n\,|f(n) - f(n-1)| \geq K. \tag{4}$$

Then there exist constants $M_k \geq m_k > 0$ and $t_0 > 0$ such that for any $|t| > t_0$,

$$m_k \frac{|t|}{(f(t))^k} \leq \left\|e^{A_1 t} A_1^{-k}\right\| \leq M_k \frac{|t|}{(f(t))^k} \tag{5}$$

where $f(t)$ is any continuous and even extension of the sequence $\{f(n)\}_{n\in\mathbb{N}}$ to $\mathbb{R}$.

The proof of this theorem uses the discrete form of the classical Hardy inequality for $p = 2$ several times. Note that in the case $k = 0$ the two-sided estimate (5) turns into (3).

Theorem 1 means that these C_0-groups $\left\{e^{A_1 t}\right\}_{t\in\mathbb{R}}$ have sub-linear growth on $D(A^k)$ and we know an exact hierarchy of growth, i.e. as k increases it turns out that the rate of growth strictly decreases. And this fact, in its turn, implies the sub-linear growth of the classical (starting from $D(A_1)$) and all more regular (starting from $D(A_1^k)$, $k \geq 2$) solutions of the corresponding Cauchy problems for the abstract linear evolution equations

$$\dot{x} = A_1 x, \;\; x(0) = x_0,$$

on the space H_1.

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Differential equation $ay' = by^m$ in the ring of power series over some commutative rings

Roman Skurikhin (*Kharkiv, Ukraine*)

Let R be a commutative unique factorization domain (UFD) with characteristic zero, meaning that R is a commutative ring with unity in which every non-zero non-unit element can be uniquely factored into irreducible elements (up to units and order of factors), and $n \cdot 1_R \neq 0$ for all $n \in \mathbb{N}$. For example, R can be the ring of integers $\mathbb{Z}$.

Let $a, b, c_0 \in R$, $a, b \neq 0$. Consider the Cauchy problem in the ring of formal power series with coefficients in R:

$$ay' = by^m, \quad y(0) = c_0.$$

We seek a solution in the form of a formal power series $R[[x]]$:

$$y = \sum_{k=0}^{\infty} c_k x^k, \quad c_k \in R.$$

Theorem 1. *The existence of a solution in $R[[x]]$ depends on the value of m as follows:*

1. *If $m = 1$, consider the equation $y' = y$, then there is a non-zero solution if and only if R contains all rational numbers.*

2. *If $m = 2$, Cauchy problem $ay' = by, y(0) = c_0$ has a solution in $R[[x]]$ if and only if*
$$a \mid b\, c_0.$$

3. *If $m \geq 3$, let $d = m - 1$. For each prime element $\pi \in R$ lying over a rational prime $p \mid d$ (that is, $\pi \mid p$ in R), denote by $v_\pi(\cdot)$ the corresponding π-adic valuation in R. Then define*
$$r = \prod_{\pi \mid p,\ p \mid d} \pi^{\left\lceil v_\pi(p)/(p-1) \right\rceil}.$$
Then a solution of the Cauchy problem $ay' = by, y(0) = c_0$ exists in $R[[x]]$ if and only if
$$a\, r \mid b\, c_0^{m-1}.$$

Remark 1. *In case $m = 3$, intuitively, r collects all prime divisors in R that originate from the primes p dividing d, raised to the minimal powers ensuring integrality of the coefficients in the formal solution.*

Corollary 1. *Suppose,* $\mathbb{Q} \not\subseteq R$*. Then equation* $y' = y^m$ *admits a formal power series solution with coefficients in* R *for every initial value* $y(0) = c_0 \in R$ *if and only if* $m = 2$*.*

Corollary 2. *Let* $m \geq 2$*. Then equation* $y' = y^m$ *has a non-zero formal power series solution for some initial value* $y(0) = c_0 \in R$*.*

This behavior is reminiscent of the case of the ring of formal Laurent series with negative powers. According to the so-called *Theorem "about two"* from [2], in the Laurent series setting the equation $y' = y^m$ admits a non-zero solution only when $m = 2$. Thus, in both the power series and Laurent series contexts the case $m = 2$ is exceptional, although in the power series case nonzero solutions also exist for other values of m.

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One constructive problem of controllability for LS

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In many cases, it is very difficult to find a control that solves the problem of timeoptimal control. As already noted in [1] an interesting problem is getting from arbitrary point to the given point in a finite time with control restrictions. Return sets whose points are transferable into themselves after a certain period in detail studied in [2], a lot of interesting research is also available in case of attaining into stationary point, but usefully to solve so-called "the constructive problem of controllability", which satisfies given restriction and transfers an arbitrary point to a given non-equilibrium point in a finite time using the controllability function method, which was proposed by V. I. Korobov in [3]. In [4] was carried out the problem of construction of a constrained control, which transfers a system from initial point to a given non-equilibrium point in a finite time was initiated in the paper [5]. Through the article [4] we solved this problem for arbitrary linear systems and for nonlinear systems like $\dot{x} = f(x, u)$, $x \in R^n$, $u \in \Omega = \{u \in R^r : \|u\| \leq d\} \subset R^r$, which can be reduced to linear ones, were illustrated results on model examples, discussed nuances and feature that arise in solving the equation that defines the controllability function, which have not been identified previously.

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Solving the problem of finding the optimal time to switch to another drug in oncology

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Oncologists widely recognize cancer as a complex system. Such systems are characterized by biological, chemical, and mechanical structures that change over time to support tumor growth [1]. Most cancers are treatable with surgery, radiation therapy, or chemotherapy, especially when detected early. Chemotherapy is one of the most common and effective methods of fighting cancer [2]. Despite its success, chemotherapy is not without limitations. One of the main drawbacks is the toxicity of chemotherapeutic drugs.

Let us assume that drug A, which was selected as the first-line drug, has better efficacy, as assessed by RECIST criteria, which divide patients into those with progressive disease (PD), stable disease (SD), partial response (PR), or complete response (CR) to the drug. This drug is predicted to cause the first adverse event (AE) associated with one of the human organs according to the SDISC classification after a time C_1. Another drug, B, is less effective, but when used, the same AE occurs later than when drug A is used. It is necessary to find the optimal time to switch from drug A to drug B in order to simultaneously obtain the longest possible effectiveness of drug A and to delay the occurrence of AE as long as possible.

In the first stage, we plan to use the system of equations (1) discussed in [3] as a basis for solving this problem:

$$\begin{cases} N'(t) = aN(t)(1 - bN(t)) - \alpha_1 N(t)T(t) - k_N u(t)N(t), \\ L'(t) = rN(t)T(t) - \mu L(t) - \beta_1 L(t)T(t) - k_L u(t)L(t), \\ T'(t) = cT(t)\big(1 - dT(t)\big) - \alpha_2 N(t)T(t) - \beta_2 L(t)T(t) - \\ \qquad - k_T u(t)T(t), \\ u'(t) = v - \omega u(t), \end{cases} \tag{1}$$

with initial conditions

$$N(0) = N_0 \geq 0, \quad L(0) = L_0 \geq 0, \quad T(0) = T_0 \geq 0, \quad u(0) = u_0 \geq 0.$$

The first equation of model (1) assumes that NK cells grow logistically during the period $aN(t)(1 - bN(t))$. However, they are inactivated due to interaction with tumor cells based on $-\alpha_1 N(t)T(t)$. In the second equation, CTL cells are present in the system when tumor cells are present. They are recruited by tumor cells via a linear term $rN(t)T(t)$. Furthermore, CTL cell death is a linear process $-\mu L(t)$. Furthermore, interaction with tumor cells inactivates CTL cells via $-\beta_1 L(t)T(t)$.

In the third equation, it is assumed that the tumor grows according to a logistic function as $cT(t)(1-dT(t))$. In addition, tumor cells are destroyed by both NK cells and CTL cells, which is implemented by $\alpha_2 N(t)T(t)$ and $\beta_2 L(t)T(t)$, respectively. The chemotherapeutic drug in the last equation has a constant source v and linearly disappears from the system at $-\omega u(t)$. In model (1), since the chemotherapeutic drug affects all three cell populations through mass action dynamics, and the mortality rate from the drug differs for each cell type, three different reaction coefficients were designated as k_N, k_L, k_T [3].

In the future, we plan to study this system, improve it, and validate the solution based on information obtained from clinical studies and other data published by other researchers in this field.

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Return condition and controllability into a non-equilibrium point

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This paper is devoted to the problem of null-controllability into a non-equilibrium point, and to the concept of the return condition on an interval proposed by V. I. Korobov in [1] for the linear system

$$\dot{x} = Ax + \varphi(u), \quad x \in \mathbb{R}^n, \quad u \in \Omega \subset \mathbb{R}^r.$$

The problem of null-controllability to an equilibrium point has been studied in many papers, including [2, 3, 4]. In this case, it is assumed that after reaching the origin, there exists a value of control $u = u_0$, such that $\varphi(u_0) = 0$, and the system stays at rest indefinitely.

However, one may also consider the case when such a value u_0 does not exist and the origin is not an equilibrium point. Such a problem has been considered, for example, in [1, 5, 6]. In [1] it was proven that the system is null-controllable to a non-equilibrium point, if and only if, in addition to conditions for ordinary null-controllability, the system satisfies an additional condition, called the return condition on an interval. This condition means that there exists a time interval $I = [T_0, T_0 + a]$, $a > 0$, such that for any $T \in I$ it is possible to construct a control such that trajectory starts at the origin and returns there at time T.

In the paper, we consider in detail the class of system for which the return condition is satisfied. We seek the solution in the form of a piecewise-constant control with values $u = \frac{1}{2}$ and $u = 1$, and show that the return condition on an interval is satisfied for the oscillating system

$$\begin{cases} \dot{x}_1 = kx_2, \\ \dot{x}_2 = -kx_1 + u, \end{cases} \qquad k = \overline{1, n}, \quad u \in [c, 1], \; c \leq \frac{1}{2}, \tag{1}$$

and then generalize the results for an arbitrary linear system with imaginary eigenvalues $\lambda_{2k-1,2k} = \pm i\nu_k$, $k = 1, \ldots, n$ in the case when ν_k are rational, or co-rational (that is $\frac{\nu_i}{\nu_j}$ is rational for $i, j = \overline{1, n}$).

We propose two explicit constructions of controls that ensure the return condition. First solution has n switching moments and has the form

$$u(t)=\begin{cases}\frac{1}{2}, & 0\le t\le \frac{2\pi}{n+1},\\ 1, & \frac{2\pi}{n+1}<t\le \frac{2\pi}{n+1}+\alpha,\\ \vdots & \\ 1, & n\,\frac{2\pi}{n+1}<t\le n\,\frac{2\pi}{n+1}+\alpha,\\ \frac{1}{2}, & n\,\frac{2\pi}{n+1}+a<t\le 2\pi+\alpha.\end{cases} \tag{2}$$

The second solution utilizes the symmetry of the trajectory and transforms the problem into the exponential form of trigonometric moment problem:

$$\begin{cases}\int_0^T u(t)e^{i\,t}\mathrm{d}t=0,\\ \vdots\\ \int_0^T u(t)e^{n\,i\,t}\mathrm{d}t=0,\end{cases} \tag{3}$$

and has only two switching moments, $T_1=a$ and $T_2=2\pi$ on the interval $I=[0,2\pi+a]$ for the arbitrary dimension of the initial system.

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On the Kolmogorov width as a characteristic of reachable sets of infinite-dimensional systems

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The concept of n-width, introduced by Kolmogorov in his seminal 1936 paper [1], plays a crucial role in approximation theory and numerical analysis as a quantitative measure of how well a set can be approximated by n-dimensional linear subspaces. Although natural computational applications of the Kolmogorov width to distributed-parameter systems have mostly focused on estimating residuals of finite-difference methods, its potential for characterizing controllability-type properties remains largely unexplored.

In this presentation, we establish a general result concerning the n-width of the image of a bounded set under a class of nonlinear operators acting between two Banach spaces. Based on this representation, we investigate the Kolmogorov width of reachable sets for bilinear control systems with bounded inputs. As a result, we derive an explicit inequality relating the Kolmogorov width of the reachable set to that of the set of admissible controls. Such an inequality can be used to estimate the dimension of a required finite-dimensional approximation, depending on a prescribed tolerance for approximating reachable states. Our construction is illustrated with an example of a controlled bilinear Schrödinger equation.

This work is partially based on the results presented in [2].

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[2] Zuyev A., Feng L., Benner P.: Estimates of the Kolmogorov n-width for nonlinear transformations with application to distributed-parameter control systems. IEEE Control Systems Letters 8, 1877-1882 (2024).

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